

ENERGY-MOMENTUM TENSORS FOR THE QUARK-GLUON PLASMA

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Abstract

We construct the energy-momentum tensor for the gauge fields which describe the collective excitations of the quark-gluon plasma. We rely on the description of the collective modes that we have derived in previous works. By using the conservation laws for energy and momentum, we obtain three different versions for the tensor $T^{\mu\nu}$, which are physically equivalent. We show that the total energy constructed from T^{00} is positive for any non-trivial field configuration. Finally, we present a new non-abelian solution of the equations of motion for the gauge fields. This solution corresponds to spatially uniform color oscillations of the plasma.

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1 Introduction

The low-lying excitations of the quark-gluon plasma at high temperature are collective excitations which involve the coherent behaviour of a large number of particles over typical distances and times of order $1/gT$. These excitations can be conveniently described as oscillations of self-consistent average fields to which the plasma particles couple[1]. We have shown that, at leading order in the coupling g , the Dyson-Schwinger equations for the N -point functions ($N \geq 1$) reduce to a set of coupled equations for these mean fields and for 2-point functions. The equations for the 2-point functions describe the dynamics of the hard plasma particles in the presence of soft background fields; they may be given the form of simple kinetic equations for generalized on-shell distribution functions[1].

In the present work, we investigate further the properties of the collective modes, in order to properly characterize their energy-momentum content. We restrict ourselves here to the case of *bosonic* collective excitations. Special attention has been paid recently to this problem[2, 3, 4]. An early attempt to derive the corresponding energy-momentum tensor $T^{\mu\nu}$ can be found in Ref. [2]. There, auxiliary fields are introduced in order to write a local effective action, for which the Noether construction is available. The expression for $T^{\mu\nu}$ thus obtained is rather intricate and difficult to use. Another approach is used in Ref. [3], where the plasma is coupled with a weak gravitational field. Again, a complicated form is obtained for $T^{\mu\nu}$, which makes its physical interpretation difficult. This is true, in particular, for the terms quadratic in the gauge fields A_μ , which, however, should be analogous to those corresponding to an abelian plasma. We would certainly expect the expressions appropriate for the QED plasma to be simple, easy to interpret, and closely related to the ones corresponding to classical polarizable media. Another difficulty with the derivation proposed in [3] is that the resulting field energy, $P^0 \equiv \int d^3x T^{00}$, appears to be negative even for some simple, plane-wave, field configurations, thus casting doubts on the stability of the equilibrium state. It is, however, verified in [3] that P^0 is positive for the abelian-like plasma normal modes. A similar, but more general, conclusion emerges from Ref. [4], where a Hamiltonian for the soft gauge fields is constructed in a completely different approach[5], by exploiting the analogy between the effective action for the collective excitations[6, 7, 1] and the eikonal of a Chern-Simons theory. The resulting functional is positive when evaluated for gauge fields which satisfy the non-abelian equations of motion without sources, that is, for general, non-abelian, collective excitations.

As we shall show, our approach overcomes most of the difficulties mentioned above and has the advantage of offering a transparent physical interpretation. We study a general

configuration of gauge fields which may be induced by an appropriate external color source j_a^μ . By using the conservation laws for energy and momentum, we construct the tensor $T^{\mu\nu}$ for such gauge fields. In this procedure, we rely on the explicit expressions for the retarded non-abelian induced current j_μ^{ind} that we have obtained in previous works[1]. The essential aspects of our approach are presented in Section 2.

Of course, the construction of $T^{\mu\nu}$ from conservation laws does not give a unique answer, and, in fact, we shall write down three different expressions, which are physically equivalent:

- (i) The first expression, derived in Sec.3.1. and referred simply as $T^{\mu\nu}$ generalizes well-known expressions for classical dielectric media (see, e.g., Refs.[15]), to which it reduces in the abelian case. The corresponding energy density T^{00} is *manifestly positive* for arbitrary gauge fields. The same is therefore true for the corresponding field energy P^0 .
- (ii) The second version for the energy-momentum tensor, referred as $\tilde{T}^{\mu\nu}$, is derived in Sec.3.2. It is equivalent, but slightly simpler, to the expression obtained by Weldon in Ref. [2], and differs from it only through a “superpotential”[13]. The components $\tilde{T}^{\mu i}$ ($i = 1, 2, 3$) of this new tensor have a simpler non-local structure than $T^{\mu i}$.
- (iii) Finally, the third version, derived in Sec.3.3 and denoted as $T_{\text{sym}}^{\mu\nu}$, is traceless and symmetric. This allows us, in particular, to construct the gauge field angular momentum.

In Section 4, we show that the expressions that we have obtained reduce to familiar ones when specialized to the case of abelian or quasi-abelian plasmas (i.e., in the weak field limit). In Section 5, we briefly discuss new, truly non-abelian, solutions of the equations of motion, which correspond to global color oscillations. The main conclusions are summarized in Section 6.

2 Soft fields and hard particles

We consider an ultrarelativistic quark-gluon plasma close to thermal equilibrium, at a temperature $T = 1/\beta$ and zero chemical potential. We use natural units, $\hbar = c = 1$, and Minkovski metric. We consider a $SU(N)$ gauge theory with N_f flavors of quarks. The generators of the gauge group in different representations are taken to be Hermitian and traceless. They are denoted by t^a and T^a , respectively, for the fundamental and the adjoint representation, and are normalized such that $\text{tr}(t^a t^b) = (1/2)\delta^{ab}$ and $\text{tr}(T^a T^b) = N\delta^{ab}$. It follows that $(T^a)_{bc} = -if^{abc}$, where f^{abc} are the structure constants of the group: $[T^a, T^b] = if^{abc}T^c$. We use, without distinction, upper and lower positions for the color indices.

As alluded to in the introductory section, we are studying the response of the plasma to a classical, external, color current $j_\mu \equiv j_\mu^a T^a$. In the absence of the external source, the plasma is assumed to be in thermal equilibrium and the expectation values of the fields vanish. The interaction with the external current is adiabatically turned on from time $t_0 \rightarrow -\infty$, and the gauge fields acquire then nontrivial expectation values, denoted by $A_\mu^a(x)$.

We assume the external perturbation to be *weak* and *slowly varying*. Then, the induced average fields have Fourier components with only *soft* momenta $P \sim gT$, and their amplitude is constrained so that $F_{\mu\nu} \lesssim gT^2$, or, equivalently, $A_\mu \lesssim T[1]$. The latter limitation on the amplitude of the field oscillations ensures in particular the consistency of the soft covariant derivatives: if $A_\mu \sim T$, then $gA_\mu \sim gT$ is of the same order as the derivative of a slowly varying quantity, $i\partial_\mu \sim gA_\mu$. Occasionally, we shall specialize our results to the case of weaker fields, for which it is consistent to replace covariant derivatives by ordinary ones; then, A_μ behaves as an abelian gauge field and the equations are linear. This weak field limit will be referred below as “the abelian regime”. (For a QED plasma, the equations remain linear even for fields as strong as allowed, i.e., for $A_\mu \sim T$.)

The gauge fields satisfy the generalized Maxwell equation

$$[D^\nu, F_{\nu\mu}(x)]^a = j_\mu^a(x) + j_\mu^{ind a}(x), \quad (2.1)$$

where $D_\mu = \partial_\mu + igA_\mu(x)$, ($A_\mu \equiv A_\mu^a T^a$), and $F_{\mu\nu} = [D_\mu, D_\nu]/(ig) = F_{\mu\nu}^a T^a$. The external source $j_\mu^a(x)$ is to be considered as a formal device to generate arbitrary gauge field configurations. In leading order, the induced current j_μ^{ind} is proportional to the fluctuations of the average phase-space color densities of plasma constituents. These fluctuations are described by color matrices which we denote by $\delta n_\pm(\mathbf{k}, x) \equiv \delta n_\pm^a(\mathbf{k}, x) t^a$ and $\delta N(\mathbf{k}, x) \equiv \delta N^a(\mathbf{k}, x) T^a$ for quarks, antiquarks and gluons, respectively. Then

$$\begin{aligned} j_\mu^{ind a}(x) &= g \int \frac{d^3k}{(2\pi)^3} v_\mu \text{Tr} \left\{ 2 N_f t^a [\delta n_+(\mathbf{k}, x) - \delta n_-(\mathbf{k}, x)] + 2 T^a \delta N(\mathbf{k}, x) \right\} \\ &= g \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu}{k} \left\{ N_f [\delta n_+^a(\mathbf{k}, x) - \delta n_-^a(\mathbf{k}, x)] + 2 N \delta N^a(\mathbf{k}, x) \right\}, \end{aligned} \quad (2.2)$$

where $k^0 = |\mathbf{k}|$ and the factors of 2 account for the spin degrees of freedom. The quantities δn_\pm and δN are determined by the following kinetic equations[1]:

$$[v \cdot D_x, \delta n_\pm(\mathbf{k}, x)] = \mp g \mathbf{v} \cdot \mathbf{E}(x) \frac{dn_k}{dk}, \quad (2.3)$$

$$[v \cdot D_x, \delta N(\mathbf{k}, x)] = -g \mathbf{v} \cdot \mathbf{E}(x) \frac{dN_k}{dk}. \quad (2.4)$$

Here, $v^\mu \equiv (1, \mathbf{v})$, where $\mathbf{v} \equiv \mathbf{k}/k$ is the velocity of the hard particle ($k \equiv |\mathbf{k}|$), $E^i \equiv F^{i0}$ is the chromoelectric field. In the r.h.s., $N_k \equiv 1/(\exp(\beta k) - 1)$ and $n_k \equiv 1/(\exp(\beta k) + 1)$ denote, respectively, equilibrium boson and fermion occupation factors.

Eqs. (2.3) and (2.4) generalize the Vlasov equation to non-abelian plasmas. They are covariant under a local gauge transformation of the mean fields $A_\mu^a(x)$; therefore, the fluctuations $\delta n_\pm^a(\mathbf{k}, x)$ and $\delta N^a(\mathbf{k}, x)$ transform like vectors in the adjoint representation. The gauge symmetry requires the presence of covariant derivatives, which makes these equations non-linear with respect to A_μ^a . If we were to solve these equations for a fixed $v^\mu \equiv (1, \mathbf{v})$, we could get rid of the non-linear terms by choosing the *axial gauge* $v^\mu A_\mu^a(x) = 0$. In this gauge, $(v \cdot D)^{ab} = \delta^{ab} v \cdot \partial$ and $\mathbf{v} \cdot \mathbf{E}^a(x) = -\mathbf{v} \cdot (\partial_0 \mathbf{A}^a + \nabla A_0^a)$, as for abelian fields. However, in calculating the induced current (2.2) we have to integrate over all the directions of \mathbf{v} . It is therefore necessary to solve eqs. (2.3)–(2.4) in an arbitrary gauge. This can be done with the help of the retarded Green's functions for the covariant line derivative $v \cdot D_x$, defined by

$$i(v \cdot D_x)_{ac} G_{ret}^{cb}(x, y; v) = \delta^{ab} \delta^{(4)}(x - y), \quad G_{ret}(x, y; v) = 0 \text{ for } x_0 < y_0, \quad (2.5)$$

and which has the following expression

$$G_{ret}(x, y; v) = -i \theta(x^0 - y^0) \delta^{(3)}(\mathbf{x} - \mathbf{y} - \mathbf{v}(x^0 - y^0)) U(x, y). \quad (2.6)$$

Here, $U(x, y)$ is the parallel transporter along the straight line γ joining x and y ,

$$U(x, y) = P \exp\{-ig \int_\gamma dz^\mu A_\mu(z)\}. \quad (2.7)$$

The solution of eq. (2.3) is then

$$\delta n_\pm^a(\mathbf{k}, x) = \mp g \frac{dn_k}{dk} \int_0^\infty du U_{ab}(x, x - vu) \mathbf{v} \cdot \mathbf{E}^b(x - vu), \quad (2.8)$$

with a similar expression for $\delta N^a(\mathbf{k}, x)$. From eqs. (2.2) and (2.8), we readily derive the induced current

$$j_{\mu a}^{ind}(x) = 3\omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu \int_0^\infty du U_{ab}(x, x - vu) \mathbf{v} \cdot \mathbf{E}^b(x - vu), \quad (2.9)$$

where ω_p is the *plasma frequency*, $\omega_p^2 \equiv (2N + N_f)g^2 T^2/18$. The integral $\int d\Omega$ runs over all the directions of the unit vector \mathbf{v} . It can be easily verified that $j_{\mu a}^{ind}(x)$ is covariantly conserved,

$$[D^\mu, j_\mu^{ind}(x)] = 0. \quad (2.10)$$

It follows that the external current is constrained by a similar equation, $[D^\mu, j_\mu(x)] = 0$.

Eq. (2.9) for the induced current summarizes all the information about the collective motion of the hard particles in the presence of the soft gauge fields. By using it in the r.h.s. of eq. (2.1), one obtains a generalization of Maxwell's equations in a polarizable medium. Besides, the induced current acts as a *generating functional for all the (retarded) amplitudes between soft gauge fields*. For instance, the polarization tensor for soft gluons is

$$\Pi_{\mu\nu}^{ab}(x, y) = \frac{\delta j_{\mu a}^{ind}(x)}{\delta A_b^\nu(y)} \Big|_{A=0}, \quad (2.11)$$

which gives (with $P^\mu = (p^0, \mathbf{p})$)

$$\Pi_{\mu\nu}^{ab}(P) = 3 \omega_p^2 \delta^{ab} \left\{ -\delta_\mu^0 \delta_\nu^0 + p^0 \int \frac{d\Omega}{4\pi} \frac{v_\mu v_\nu}{v \cdot P + i\eta} \right\}. \quad (2.12)$$

This is the well-known “hard thermal loop” for the gluon self-energy[8, 9, 10, 12]. In the weak field limit, the tensor $\Pi_{\mu\nu}$ fully characterizes the dielectric properties of the plasma. For strong fields, $A_\mu \sim T$, all N -gluons amplitudes with $N \geq 2$ contribute to the induced current (2.9) at the same order in g [11, 12, 1].

The induced current may be given a different functional form which will be used later. We start by defining a new function, $W^\mu(x; v)$, as the solution to

$$[v \cdot D_x, W^\mu(x; v)] = F^{\mu\rho}(x) v_\rho, \quad (2.13)$$

which vanishes when $x_0 \rightarrow -\infty$. We have $W^\mu = W_a^\mu T^a$, with

$$W_a^\mu(x; v) = \int_0^\infty du U_{ab}(x, x - vu) F_b^{\mu\rho}(x - vu) v_\rho, \quad (2.14)$$

and the induced current (2.9) can be rewritten as

$$j_{\mu a}^{ind}(x) = 3 \omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu W_a^0(x; v). \quad (2.15)$$

In the case of an abelian plasma, eW^μ has a simple physical interpretation: it represents the 4-momentum transferred by the electromagnetic field to a particle of charge e and velocity \mathbf{v} , moving along a straight line from time $t_0 \rightarrow -\infty$ to x_0 . (The transferred momentum $\sim gT$ is small compared to the hard momentum $k \sim T$, so that the particle is not significantly deviated by the Lorentz force.) From eq. (2.14) it is obvious that

$$v_\mu W_a^\mu(x; v) = 0. \quad (2.16)$$

The functions $W_a^\mu(x; v)$ are homogeneous of degree zero as functions of v^μ , i.e.,

$$v^\nu \frac{\partial W_a^\mu(x; v)}{\partial v^\nu} = 0. \quad (2.17)$$

(In taking derivatives with respect to v^μ , as in eq. (2.17), we consider the four components of v^μ as independent variables, and set $v^\mu = (1, \mathbf{v})$ only after derivation.) This is obvious from the defining equation (2.13), and can be also verified on eq. (2.14) by using the identity

$$v^\nu \frac{\partial}{\partial v^\nu} f(x - vu) = -u (v \cdot \partial_x) f(x - vu). \quad (2.18)$$

We prove now that the expressions (2.9) or (2.15) for the induced current are equivalent to the following one

$$j_{\mu a}^{ind}(x) = \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \left[W_\mu^a(x; v) + v_\mu \frac{\partial W_a^\rho(x; v)}{\partial v^\rho} \right], \quad (2.19)$$

which also appears, with slightly different notations, in the last of Refs.[1] (as eq. (C.15)). To show the equivalence, we note first that eq. (2.17) implies

$$\frac{\partial W_a^\rho(x; v)}{\partial v^\rho} = \frac{\partial W_a^i(x; v)}{\partial v^i} - \left(v^i \frac{\partial}{\partial v^i} \right) W_a^0(x; v). \quad (2.20)$$

Then, we use eq. (2.16) to replace $W_a^0(x; v)$ by $\mathbf{v} \cdot \mathbf{W}_a(x; v)$ in (2.20), and obtain

$$\frac{\partial W_a^\rho(x; v)}{\partial v^\rho} = W_a^0(x; v) + \left(\frac{\partial}{\partial v^i} - \frac{\partial}{\partial v^j} \frac{v^i v^j}{\mathbf{v}^2} \right) W_a^i(x; v). \quad (2.21)$$

The second term in the r.h.s. vanishes upon angular integration, so that

$$\int \frac{d\Omega}{4\pi} \frac{\partial W_a^\rho(x; v)}{\partial v^\rho} = \int \frac{d\Omega}{4\pi} W_a^0(x; v). \quad (2.22)$$

This proves the equivalence between the time components ($\mu = 0$) of eqs. (2.15) and (2.19). The corresponding proof for the spatial components is similar.

3 Energy-momentum tensor from conservation laws

The energy-momentum tensor $T_{\mu\nu}(x)$ of the collective excitation satisfies the following conservation law

$$\partial_\mu T^{\mu\nu}(x) = -F_a^{\nu\mu}(x) j_\mu^a(x), \quad (3.1)$$

for fields satisfying (2.1). This equation may be viewed as the thermal average of the conservation law formally satisfied by the corresponding field operators in the presence of the external perturbation j_μ^a . (Since no ultraviolet divergence appears at the level of the

present approximation[12, 1], we can safely ignore all the complications concerning the finiteness of $T_{\mu\nu}(x)$ in perturbation theory[13].) Note, however, that we shall not derive $T_{\mu\nu}(x)$ by directly evaluating the expectation value of some local composite operator (e.g., the canonical energy-momentum tensor associated to the QCD lagrangian), but, rather, by integrating the conservation law (3.1). This is possible because, in the present approximation, all the dynamical information is contained in the equation of motion (2.1) for the gauge fields, and we know, from eqs. (2.9) or (2.19), the explicit expression of the induced current in terms of the fields.

In order to construct the tensor $T^{\mu\nu}$, it is convenient to separate the standard Yang-Mills contribution, by writing

$$T^{\mu\nu}(x) \equiv T_{YM}^{\mu\nu}(x) + \Theta^{\mu\nu}(x), \quad (3.2)$$

with

$$T_{YM}^{\mu\nu}(x) \equiv -F_a^{\mu\rho}(x) F_{a\rho}^\nu(x) + \frac{1}{4} g^{\mu\nu} F_a^{\alpha\beta}(x) F_{\alpha\beta}^a(x). \quad (3.3)$$

The tensor (3.3) satisfies the structural equation

$$\partial_\mu T_{YM}^{\mu\nu}(x) = - [D^\mu, F_{\mu\rho}(x)]^a F_a^{\nu\rho}(x). \quad (3.4)$$

In deriving this equation, the following identity is useful:

$$\partial_\mu (A^a B^a) = [D_\mu, A]^a B^a + A^a [D_\mu, B]^a. \quad (3.5)$$

We use now the equations of motion (2.1) for $F^{\mu\nu}$ to transform eq. (3.4) into

$$\partial_\mu T_{YM}^{\mu\nu}(x) = - F_a^{\nu\rho}(x) (j_\rho^a(x) + j_\rho^{ind a}(x)). \quad (3.6)$$

By comparing eqs. (3.1), (3.2) and (3.6), we conclude that $\Theta^{\mu\nu}(x)$ satisfies

$$\partial_\mu \Theta^{\mu\nu}(x) = F_a^{\nu\mu}(x) j_\mu^{ind a}(x). \quad (3.7)$$

In the next sections we shall use either eq. (2.15) or eq. (2.19) for j_μ^{ind} in order to write the r.h.s. of eq. (3.7) as a total derivative of some gauge-invariant functional of the fields, and identify in this way $\Theta^{\mu\nu}(x)$.

The tensor $T_{\mu\nu}$ thus defined will satisfy, by construction, the conservation law (3.1), and will vanish for large negative times. Accordingly, the 4-momentum constructed from it,

$$P^\nu(t) \equiv \int d^3x T^{0\nu}(t, \mathbf{x}), \quad (3.8)$$

will coincide, at any time, with the correct 4-momentum of the collective excitation, which is obtained by integrating the conservation law

$$\partial_0 P^\nu(t) = - \int d^3x F_a^{\nu\mu}(x) j_\mu^a(x). \quad (3.9)$$

(In principle, this equation determines $P^\nu(t)$, given the initial condition $P^\nu(t \rightarrow -\infty) = 0$. Recall that the external sources and the average fields are assumed to vanish at $t \rightarrow -\infty$). Of course, the tensor $T_{\mu\nu}$ thus obtained is not unique, since any two tensors which would differ by a divergence-free piece, and lead to the same P^μ , would be equally acceptable. We shall indeed present in this section three different versions of the energy-momentum tensor.

3.1 A positive definite energy density

Eq. (2.15) for j_μ^{ind} allows us to write the r.h.s. of eq. (3.7) as

$$\begin{aligned} F_a^{\nu\mu}(x) j_\mu^{ind a}(x) &= 3 \omega_p^2 \int \frac{d\Omega}{4\pi} F_a^{\nu\mu}(x) v_\mu W_a^0(x; v) \\ &= 3 \omega_p^2 \int \frac{d\Omega}{4\pi} [v \cdot D_x, W^\nu(x; v)]^a W_a^0(x; v), \end{aligned} \quad (3.10)$$

where eq. (2.13) for $W^\nu(x; v)$ has been used in the second line. For $\nu = 0$, this can be easily written as a total derivative,

$$\begin{aligned} \mathbf{E}^a(x) \cdot \mathbf{j}_a^{ind}(x) &= 3 \omega_p^2 \int \frac{d\Omega}{4\pi} [v \cdot D_x, W^0(x; v)]^a W_a^0(x; v) \\ &= \partial_\mu \Theta^{\mu 0}(x), \end{aligned} \quad (3.11)$$

with the definition

$$\Theta^{\mu 0}(x) \equiv \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} v^\mu W_a^0(x; v) W_a^0(x; v), \quad (3.12)$$

where the identity (3.5) was used. After adding $T_{YM}^{\mu 0}(x)$, eq. (3.3), we end up with the following expressions for the energy density

$$T^{00}(x) = \frac{1}{2} (\mathbf{E}^a(x) \cdot \mathbf{E}^a(x) + \mathbf{B}^a(x) \cdot \mathbf{B}^a(x)) + \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} W_a^0(x; v) W_a^0(x; v), \quad (3.13)$$

and for the energy flux density, or Poynting vector, $S^i \equiv T^{i0}$,

$$\mathbf{S}(x) = \mathbf{E}^a(x) \times \mathbf{B}^a(x) + \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \mathbf{v} W_a^0(x; v) W_a^0(x; v). \quad (3.14)$$

The chromomagnetic field components are, as usual, $B_a^i(x) \equiv -(1/2) \epsilon^{ijk} F_a^{jk}(x)$.

The expression (3.13) for $T^{00}(x)$ is manifestly positive semidefinite and the same is obviously true, at any time t , for the excitation energy $\mathcal{E}(t)$,

$$\mathcal{E}(t) \equiv \int d^3x T^{00}(t, \mathbf{x}) = \mathcal{E}_{YM}(t) + \mathcal{W}(t). \quad (3.15)$$

Note that

$$\mathcal{E}_{YM}(t) \equiv \int d^3x \frac{1}{2} (\mathbf{E}^a(x) \cdot \mathbf{E}^a(x) + \mathbf{B}^a(x) \cdot \mathbf{B}^a(x)) \quad (3.16)$$

also involves medium effects, since $\mathbf{E}^a(x)$ and $\mathbf{B}^a(x)$ are the *total* gauge fields, including the plasma polarization. The second piece of (3.15),

$$\mathcal{W}(t) \equiv \frac{3}{2} \omega_p^2 \int d^3x \int \frac{d\Omega}{4\pi} W_a^0(x; v) W_a^0(x; v), \quad (3.17)$$

may be interpreted as the *polarization energy* of the plasma, that is, the energy transferred by the gauge fields to the plasma constituents as mechanical work of the chromoelectric field. This becomes more obvious if eq. (3.11) is used to reexpress $\mathcal{W}(t)$ as

$$\mathcal{W}(t) = \int_{-\infty}^t dt' \int d^3x \mathbf{E}^a(t', \mathbf{x}) \cdot \mathbf{j}_a^{ind}(t', \mathbf{x}). \quad (3.18)$$

One can easily verify – by explicitly performing the integral over t' in the r.h.s. – that eq. (3.18) leads indeed to the expression (3.17) for $\mathcal{W}(t)$ if eq. (2.15) for the induced current is used.

The positivity of the polarization energy (3.17), and hence of the total excitation energy (3.15), reflects the stability of the equilibrium state with respect to long wavelength color fluctuations.

We briefly discuss now the other components of the energy-momentum tensor $\Theta^{\mu\nu}(x)$. We can write eq. (3.7) as

$$\partial_\mu \Theta^{\mu\nu}(x) = \int \frac{d\Omega}{4\pi} F_a^{\nu\mu}(x) \mathcal{J}_\mu^a(x; v), \quad (3.19)$$

where, according to eq. (2.15), $\mathcal{J}_a^\mu(x; v) \equiv 3\omega_p^2 v^\mu W_a^0(x; v)$. We denote by $\Pi^\nu(x; v)$ the solution to

$$(v \cdot \partial_x) \Pi^\nu(x; v) = F_a^{\nu\rho}(x) \mathcal{J}_\rho^a(x; v), \quad (3.20)$$

which vanishes for $x_0 \rightarrow -\infty$, that is,

$$\Pi^\nu(x; v) = \int_0^\infty d\tau F_a^{\nu\rho}(x - v\tau) \mathcal{J}_\rho^a(x - v\tau; v). \quad (3.21)$$

It is then easy to see that

$$\Theta^{\mu\nu}(x) = \int \frac{d\Omega}{4\pi} v^\mu \Pi^\nu(x; v) \quad (3.22)$$

satisfies eq. (3.19). For $\nu = 0$, the integral over τ in (3.21) can be easily performed, with the result $\Pi^0(x; v) = (3/2) \omega_p^2 W_a^0(x; v) W_a^0(x; v)$. By inserting this result in eq. (3.22), we recover our previous expression for $\Theta^{\mu 0}(x)$, eq. (3.12). The physical content of the function $\Pi^\nu(x; v)$ is transparent: it represents the density of the total 4-momentum transferred from the fields to the hard particles with velocity \mathbf{v} from time $\rightarrow -\infty$ to x_0 . In particular, $\Theta^{0\nu}(x)$ is simply the angular average of $\Pi^\nu(x; v)$ over the possible orientations of \mathbf{v} .

The expression (3.22) for $\Theta^{\mu\nu}(x)$, though formally simple, involves generally a double integral along the hard particle trajectory (see eqs. (3.21) and (2.14)). We shall construct in the following section a different expression for the energy-momentum tensor, which involves a single integral along the trajectory.

3.2 Another expression for $T^{\mu\nu}$

We use now the second version for the induced color current $j_{\mu a}^{ind}$, eq. (2.19), in order to construct another energy-momentum tensor, $\tilde{T}^{\mu\nu}(x) \equiv T_{YM}^{\mu\nu}(x) + \tilde{\Theta}^{\mu\nu}(x)$, different from the one obtained in Sec.3.1, but physically equivalent.

When using eq. (2.19) for $j_{\mu a}^{ind}$, the r.h.s. of eq. (3.7) reads

$$F_a^{\nu\mu}(x) j_{\mu}^{ind a}(x) = \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \left\{ F_a^{\nu\mu}(x) W_\mu^a(x; v) + F_a^{\nu\mu}(x) v_\mu \frac{\partial W_a^\rho(x; v)}{\partial v^\rho} \right\}. \quad (3.23)$$

In the first term inside the brackets, we express $F_a^{\nu\mu}(x)$ in terms of the functions $W_a^\mu(x; v)$, using the identity

$$[D^\nu, W^\mu(x; v)] - [D^\mu, W^\nu(x; v)] = F^{\mu\nu}(x) + ig [W^\mu(x; v), W^\nu(x; v)]. \quad (3.24)$$

(To prove this identity, act on eq. (2.13) with the covariant derivative D^ν . Then, consider the similar equation where the indices μ and ν have been interchanged, and take the difference of the two equations. The identity (3.24) follows after some repeated use of the Jacobi identity.) We obtain

$$F_a^{\nu\mu}(x) W_\mu^a(x; v) = [D^\mu, W^\nu(x; v)]^a W_\mu^a(x; v) - \frac{1}{2} \partial^\nu (W \cdot W), \quad (3.25)$$

with the notation $W \cdot W \equiv W_\mu^a(x; v) W_\mu^a(x; v)$. For the second term in the r.h.s. of (3.23), we use eq. (2.13) for $W_\mu^a(x; v)$ to write

$$F_a^{\nu\mu} v_\mu \frac{\partial W_a^\rho}{\partial v^\rho} = [v \cdot D_x, W^\nu]^a \frac{\partial W_a^\rho}{\partial v^\rho}$$

$$= (v \cdot \partial_x) \left(W_a^\nu \frac{\partial W_a^\rho}{\partial v^\rho} \right) - W_a^\nu \left[v \cdot D_x, \frac{\partial W_a^\rho}{\partial v^\rho} \right]^a. \quad (3.26)$$

By using eq. (2.13) once again, we see that

$$\frac{\partial}{\partial v^\rho} [v \cdot D_x, W^\rho(x; v)]^a = \frac{\partial}{\partial v^\rho} (F_a^{\rho\mu}(x) v_\mu) = 0, \quad (3.27)$$

so that

$$F_a^{\nu\mu} v_\mu \frac{\partial W_a^\rho}{\partial v^\rho} = \partial_\mu \left(v^\mu W_a^\nu \frac{\partial W_a^\rho}{\partial v^\rho} \right) + W_a^\nu [D^\mu, W_\mu]^a. \quad (3.28)$$

By adding together eqs. (3.25) and (3.28), we finally obtain

$$F_a^{\nu\mu}(x) j_\mu^{inda}(x) = \partial_\mu \tilde{\Theta}^{\mu\nu}(x), \quad (3.29)$$

with

$$\tilde{\Theta}^{\mu\nu}(x) \equiv \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \left\{ W_a^\mu W_a^\nu + v^\mu W_a^\nu \frac{\partial W_a^\rho}{\partial v^\rho} - \frac{1}{2} g^{\mu\nu} (W \cdot W) \right\}. \quad (3.30)$$

The polarization energy density associated to (3.30), which is

$$\tilde{\Theta}^{00}(x) = \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \left\{ W_a^0 W_a^0 + W_a^0 \frac{\partial W_a^\rho}{\partial v^\rho} - \frac{1}{2} (W \cdot W) \right\}, \quad (3.31)$$

is different from the previous expression in Sec. 4, eq. (3.12). In particular, the expression above is *not* manifestly positive. One advantage of the new tensor above is that eq. (3.30) offers, for the components (μ, i) , a simpler non-local structure than eq. (3.22).

It is possible to show that the energy-momentum tensor $\tilde{T}^{\mu\nu}(x)$ is physically equivalent to that proposed by Weldon in Ref.[2]. Indeed, the difference between the two tensors is the total derivative of an antisymmetric tensor of rank 3. To be specific, we have $\tilde{T}^{\mu\nu}(x) - T_W^{\mu\nu}(x) = \partial_\rho X^{\rho\mu\nu}(x)$, with

$$X^{\rho\mu\nu}(x) \equiv \frac{3}{2} \omega_p^2 \int \frac{d\Omega}{4\pi} \left\{ (v^\rho \mathcal{V}_a^\mu(x; v) - v^\mu \mathcal{V}_a^\rho(x; v)) W_a^\nu(x; v) \right\}, \quad (3.32)$$

and $X^{\rho\mu\nu}(x) = -X^{\mu\rho\nu}(x)$. Here, $T_W^{\mu\nu}(x)$ denotes Weldon's tensor and the functions $\mathcal{V}_a^\mu(x; v)$ satisfy, by definition, the following equation:

$$[v \cdot D_x, \mathcal{V}^\mu(x; v)]^a = W_a^\mu(x; v). \quad (3.33)$$

Eq. (3.32) is obtained by first transforming Weldon's original expression with the help of the identity (3.24).

The tensors $\Theta^{\mu\nu}$ and $\tilde{\Theta}^{\mu\nu}$ constructed up to now are not symmetric, which is not surprising since the indices μ and ν enter disymmetrically in the conservation law (3.7), on which our derivation is based.

3.3 A traceless and symmetric tensor

We construct now a symmetric tensor, starting again from eq. (3.7), but using in its r.h.s. the expression (2.2) for the induced current, which we write as

$$j_\mu^{inda}(x) = g \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu}{k} \delta f^a(\mathbf{k}, x), \quad (3.34)$$

with

$$\delta f^a(\mathbf{k}, x) \equiv N_f \left[\delta n_+^a(\mathbf{k}, x) - \delta n_-^a(\mathbf{k}, x) \right] + 2N \delta N^a(\mathbf{k}, x). \quad (3.35)$$

The r.h.s. of eq. (3.7) reads then

$$F_a^{\nu\mu}(x) j_\mu^{inda}(x) = g \int \frac{d^3k}{(2\pi)^3} F_a^{\nu\mu}(x) v_\mu \delta f^a(\mathbf{k}, x). \quad (3.36)$$

We use the simple identity

$$F_a^{\nu\mu}(x) v_\mu = F_a^{i\mu}(x) \frac{\partial}{\partial k^i} (v^\nu k_\mu), \quad (3.37)$$

(recall that $v^\nu = k^\nu/k^0$ and $k^0 = |\mathbf{k}|$), and an integration by parts, to rewrite (3.36) as

$$F_a^{\nu\mu}(x) j_\mu^{inda}(x) = -g \int \frac{d^3k}{(2\pi)^3} v^\nu F_a^{i\mu}(x) k_\mu \frac{\partial}{\partial k^i} \delta f^a(\mathbf{k}, x). \quad (3.38)$$

Let $\delta\epsilon(\mathbf{k}, x)$ be the solution to

$$(v \cdot \partial_x) \delta\epsilon(\mathbf{k}, x) = -g F_a^{i\mu}(x) k_\mu \frac{\partial}{\partial k^i} \delta f^a(\mathbf{k}, x), \quad (3.39)$$

which vanishes when $x_0 \rightarrow -\infty$. Then

$$\delta\epsilon(\mathbf{k}, x) = N_f \left[\delta\epsilon_+(\mathbf{k}, x) + \delta\epsilon_-(\mathbf{k}, x) \right] + 2N \delta\epsilon_g(\mathbf{k}, x), \quad (3.40)$$

with obvious notations, and eq. (3.38) becomes

$$F_a^{\nu\mu}(x) j_\mu^{inda}(x) = \partial_\mu \Theta_{\text{sym}}^{\mu\nu}(x), \quad (3.41)$$

where

$$\Theta_{\text{sym}}^{\mu\nu}(x) \equiv \int \frac{d^3k}{(2\pi)^3} v^\mu v^\nu \left\{ N_f \left[\delta\epsilon_+(\mathbf{k}, x) + \delta\epsilon_-(\mathbf{k}, x) \right] + 2N \delta\epsilon_g(\mathbf{k}, x) \right\} \quad (3.42)$$

is a *traceless symmetric* tensor. The same is therefore true for the corresponding energy-momentum tensor $T_{\text{sym}}^{\mu\nu}(x) \equiv T_{YM}^{\mu\nu}(x) + \Theta_{\text{sym}}^{\mu\nu}(x)$. Remark that, in the weak field limit,

$\delta\epsilon(\mathbf{k}, x)$ is quadratic with respect to the gauge fields (see eq. (3.39)), as opposed to $\delta f^a(\mathbf{k}, x)$ which, in the same limit, is linear in A_μ .

The symmetry of $T_{\text{sym}}^{\mu\nu}(x)$ allows us to construct a simple density for the generalized angular momentum of the gauge fields:

$$M^{\lambda\mu\nu}(x) \equiv x^\mu T_{\text{sym}}^{\lambda\nu}(x) - x^\nu T_{\text{sym}}^{\lambda\mu}(x). \quad (3.43)$$

It can be verified, by using eq. (3.1), that $M^{\lambda\mu\nu}(x)$ satisfies the correct conservation law

$$\partial_\lambda M^{\lambda\mu\nu}(x) = - \left(x^\mu F_a^{\nu\lambda}(x) - x^\nu F_a^{\mu\lambda}(x) \right) j_\lambda^a(x), \quad (3.44)$$

where one recognizes in the r.h.s. the torque applied to the system by the external current. Since $M^{\lambda\mu\nu}(x)$ vanishes when $x_0 \rightarrow -\infty$, the total angular momentum is correctly given by

$$J^{\mu\nu}(t) = \int d^3x M^{0\mu\nu}(x), \quad (3.45)$$

for any t .

The vanishing trace of $T_{\text{sym}}^{\mu\nu}(x)$ reflects the dilatation invariance of the massless tree-level QCD lagrangian. At the level of the present approximation, there is no breaking of the dilatation symmetry, neither by radiative corrections (via the regularization procedure), nor by a thermally induced mass, which is negligible for the hard particles.

4 Applications

In this section we use the general expressions derived in Sec.3 to study the energy transfer in the plasma and the energy-momentum content of particular excited states. We consider mostly the weak field, or abelian, limit, where the components of $T^{\mu\nu}$ are quadratic in the gauge potentials. We show that, in this particular limit, our expressions are the straightforward generalization of the corresponding formulae in the standard plasma physics.

Moreover, any solution of the abelian gauge field and kinetic equations can be directly imbedded in the corresponding non-abelian equations. That is, let $A_\mu(x)$ and $W^0(x; v)$ be solutions to the coupled abelian equations

$$\partial^\nu F_{\nu\mu}(x) - 3\omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu W^0(x; v) = j_\mu(x), \quad (4.1)$$

and

$$(v \cdot \partial_x) W^0(x; v) = \mathbf{v} \cdot \mathbf{E}(x). \quad (4.2)$$

Then, $A_\mu^a(x) \equiv \lambda^a A_\mu(x)$ and $W_a^0(x; v) \equiv \lambda^a W^0(x; v)$, with constant λ^a , satisfy the corresponding non-abelian equations, (2.1) and (2.13), for an external source pointing in a fixed direction in color space, $j_\mu^a(x) = \lambda^a j_\mu(x)$. This property extends the corresponding one for classical Yang-Mills equations in vacuum (see, e.g., Ref. [14]).

A slightly more general field configuration for which the general solution is still trivial corresponds to gauge fields in the Cartan subalgebra of $SU(N)$, that is, $A^i(t) \equiv \sum_s A_s^i(t) T^s$, where the indice s takes on $N-1$ values corresponding to the $N-1$ commuting generators T^s of $SU(N)$. Then, clearly, $F_s^{\mu\nu} = \partial^\mu A_s^\nu - \partial^\nu A_s^\mu$, and $F_a^{\mu\nu} = W_a^\mu = j_a^\mu = 0$ for $a \neq s$. The non-trivial components A_s^μ , W_s^μ and j_s^μ satisfy abelian-like equations, as eqs. (4.1)–(4.2) above. Hence, the problem reduces itself to $N-1$ independent abelian theories.

In subsection 4.1, we briefly discuss the case of static gauge fields. In the following subsection, we investigate the energy transfer between fields and particles and, in particular, the energy dissipation through the Landau mechanism. Finally, in subsection 4.3, we consider some interesting special field configurations, namely, time-periodic fields and abelian plasma waves.

4.1 Static field configurations

We consider a field configuration induced by a static external color density $\rho^a(\mathbf{x})$. We choose a gauge where the potentials A_μ^a are time-independent. In this case, it is useful to reexpress $W_a^0(x; v)$, eq. (2.14), as a function of the gauge potentials themselves

$$W_a^0(x; v) = -A_0^a(x) + \int_0^\infty du U_{ab}(x, x - vu) v \cdot \dot{A}^b(x - vu), \quad (4.3)$$

where $\dot{A}^a(x) \equiv \partial_0 A^a(x)$. For a static field configuration, $\dot{A}^a(x) = 0$ and

$$W_a^0(\mathbf{x}; v) = -A_a^0(\mathbf{x}). \quad (4.4)$$

It follows that the induced current (2.15) reduces to a static color distribution

$$j_{\mu a}^{ind}(\mathbf{x}) = -g_{\mu 0} 3 \omega_p^2 A_a^0(\mathbf{x}) \equiv -g_{\mu 0} \rho_a^{ind}(x). \quad (4.5)$$

Then, the gauge field equations (2.1) become simply

$$\begin{aligned} [D_i, E^i(\mathbf{x})] + 3 \omega_p^2 A^0(\mathbf{x}) &= \rho(\mathbf{x}), \\ \epsilon^{ijk} [D_j, B^k] &= i g [A^0, E^i]. \end{aligned} \quad (4.6)$$

The first equation above clearly shows the Debye screening of the static chromoelectric field, with the screening length $\lambda_D = 1/(3\omega_p^2)^{1/2}$. Note also that the chromomagnetic field induced by a static color charge is generally not zero in a non-abelian theory. Furthermore,

$$\Theta^{00}(\mathbf{x}) = \frac{3}{2} \omega_p^2 A_a^0(\mathbf{x}) A_a^0(\mathbf{x}) = -\frac{1}{2} \rho_a^{ind}(\mathbf{x}) A_a^0(\mathbf{x}). \quad (4.7)$$

In analogy with what is commonly done in abelian plasmas, (see Appendix), we define the *color polarization vector* $\mathbf{P}^a(\mathbf{x})$ by $[D^i, P^i(\mathbf{x})]^a \equiv \rho_a^{ind}(\mathbf{x})$, and construct the *color displacement vector* $\mathbf{D}^a(x) \equiv \mathbf{E}^a(x) + \mathbf{P}^a(x)$. Then, the polarization energy (3.17) can be written as

$$\mathcal{W} = \int d^3x \Theta^{00}(\mathbf{x}) = -\frac{1}{2} \int d^3x [D^i, P^i(\mathbf{x})]^a A_a^0(\mathbf{x}) = \frac{1}{2} \int d^3x \mathbf{E}^a(\mathbf{x}) \cdot \mathbf{P}^a(\mathbf{x}), \quad (4.8)$$

as familiar in electrostatics of classical dielectric media[15]. (We used here $E_a^i(\mathbf{x}) = [D^i, A^0(\mathbf{x})]^a$ for static fields, together with eq. (3.5) and an integration by parts.) The total field energy follows:

$$\mathcal{E} = \frac{1}{2} \int d^3x (\mathbf{E}^a(\mathbf{x}) \cdot \mathbf{D}^a(\mathbf{x}) + \mathbf{B}^a(\mathbf{x}) \cdot \mathbf{B}^a(\mathbf{x})). \quad (4.9)$$

For an abelian plasma, we can solve the equation

$$(-\Delta + 3\omega_p^2) A^0(\mathbf{x}) = \rho(\mathbf{x}) \quad (4.10)$$

to get the gauge field

$$A^0(\mathbf{x}) = \int \frac{d^3y}{4\pi} \frac{\rho(\mathbf{y})}{R} e^{-R/\lambda_D}, \quad (4.11)$$

where $R \equiv |\mathbf{x} - \mathbf{y}|$. Then, the field energy can be expressed in terms of the external charge, with the standard result

$$\mathcal{E} = \frac{1}{2} \int d^3x (\mathbf{E}^2(\mathbf{x}) + 3\omega_p^2 A_0^2(\mathbf{x})) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{\rho(\mathbf{p}) \rho(-\mathbf{p})}{p^2 + 3\omega_p^2}. \quad (4.12)$$

For a non-abelian plasma, the non-linear structure of the generalized Maxwell equations (4.6) does not allow such a simple elimination of the gauge fields in terms of ρ^a .

4.2 Energy transfer and Landau damping

When studying time-dependent field configurations, there is a new feature arising: the imaginary part of the polarization tensor (2.12) is non-vanishing, which indicates that

there is dissipation of the gauge field energy in the medium. To investigate this, both for abelian and non-abelian fields, we compute the rate of energy absorption by the plasma constituents (see eq. (3.18)):

$$\frac{d\mathcal{W}(t)}{dt} = \int d^3x \mathbf{E}^a(t, \mathbf{x}) \cdot \mathbf{j}_a^{ind}(t, \mathbf{x}). \quad (4.13)$$

Consider first an abelian-like plasma, in which case the general description in Sec.2 reduces to the standard Maxwell equation in a polarizable medium[16]. For simplicity, consider a time-periodic field

$$\mathbf{E}(t, \mathbf{p}) = \mathbf{E}(\mathbf{p}) e^{\eta t} \cos \omega_0 t, \quad (4.14)$$

with $\eta \rightarrow 0_+$ and both ω_0 and $p \equiv |\mathbf{p}|$ of order gT . Then, a simple calculation shows that the average energy loss of the field, that is, the expectation value of eq. (4.13) over the period $T_0 = 2\pi/\omega_0$, is proportional to the imaginary part of the *dielectric tensor*[16]:

$$\left\langle \frac{d\mathcal{W}}{dt} \right\rangle = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_0 E^i(-\mathbf{p}) \left(\text{Im } \epsilon^{ij}(\omega_0, \mathbf{p}) \right) E^j(\mathbf{p}). \quad (4.15)$$

(The definition of ϵ^{ij} , as well as its relation with the polarization tensor, are reviewed in Appendix). In our case, eqs. (A.12) and (2.12) show that

$$\text{Im } \epsilon^{ij}(\omega, \mathbf{p}) = 3\pi \frac{\omega_p^2}{\omega} \int \frac{d\Omega}{4\pi} v^i v^j \delta(\omega - \mathbf{v} \cdot \mathbf{p}), \quad (4.16)$$

and we obtain the average energy loss as

$$\left\langle \frac{d\mathcal{W}}{dt} \right\rangle = \frac{3}{2} \pi \omega_p^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d\Omega}{4\pi} \delta(\omega_0 - \mathbf{v} \cdot \mathbf{p}) \left| \mathbf{v} \cdot \mathbf{E}(\mathbf{p}) \right|^2. \quad (4.17)$$

This expression is obviously positive: on the average, the fields transfer energy to the particles. This energy dissipation is related to the *Landau damping* of the mean fields[16], that is, to the coherent transfer of energy in between the gauge fields and the hard particles which are moving in phase with the field oscillations. As the δ function clearly shows, the dissipation is kinematically allowed only for fields carrying space-like momenta (i.e., when $\omega_0 = \mathbf{v} \cdot \mathbf{p}$).

The energy given by the fields to the colored particles is ultimately supplied by the external source, that is

$$\left\langle \frac{d\mathcal{W}}{dt} \right\rangle = - \left\langle \int d^3x \mathbf{E}(x) \cdot \mathbf{j}(x) \right\rangle, \quad (4.18)$$

as shown by the conservation law (3.1) (the local piece $d\mathcal{E}_{YM}/dt$ averages to 0). Therefore, in order to compute the energy loss of a given external source, one just has to replace the electric field E^i in eq. (4.15) in terms of the current j^μ , by solving the Maxwell equation (2.1). In the abelian regime, this can be easily done by Fourier analysis, and leads to the following expression for the average energy loss of the time-periodic external current $\mathbf{j}(\mathbf{p}) \cos \omega_0 t$:

$$\left\langle \frac{d\mathcal{W}}{dt} \right\rangle = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_0 \left\{ \frac{|\rho(\mathbf{p})|^2}{p^2} \text{Im} \frac{1}{\epsilon_l(\omega_0, p)} + \frac{|\mathbf{j}_t(\mathbf{p})|^2}{\omega_0^2} \text{Im} \frac{1}{\epsilon_t(\omega_0, p) - p^2/\omega_0^2} \right\}. \quad (4.19)$$

Here, $\omega_0 \rho(\mathbf{p}) = -i\mathbf{p} \cdot \mathbf{j}(\mathbf{p})$, as required by the current conservation, while $\mathbf{j}_t(\mathbf{p})$ is the transverse component of $\mathbf{j}(\mathbf{p})$, i.e., $j_t^i(\mathbf{p}) = (\delta^{ik} - \hat{p}^i \hat{p}^k) j^k(\mathbf{p})$. The longitudinal and transverse pieces of ϵ^{ij} are defined in Appendix (eqs. (A.13)–(A.15)).

Clearly, eq. (4.19) also holds for non-abelian field configurations such that the space-time and the color degrees of freedom decouple, that is, for the imbedded abelian solutions to eq. (2.1) with a color source $\mathbf{j}^a(t, \mathbf{p}) = \lambda^a \mathbf{j}(t, \mathbf{p})$, and constant λ^a . (The only modification consists in a supplementary factor $\lambda^a \lambda^a$ in the previous equation.) An expression similar to (4.19) has been used in Refs. [18, 19] in order to compute the energy loss in the plasma of an external parton, which is assimilated to a classical point-like colored particle, moving with constant velocity \mathbf{v} , and therefore giving rise to a current

$$\mathbf{j}^a(\omega, \mathbf{p}) = 2\pi \lambda^a \mathbf{v} \delta(\omega - \mathbf{v} \cdot \mathbf{p}). \quad (4.20)$$

Clearly, the energy loss calculated according to eq. (4.19) accounts only for the energy transferred by the external source to the hard particles, through its coupling to the collective modes. This is indeed the dominant process for long wavelength and low frequency excitations. However, these conditions are not satisfied for the whole range of momenta involved in the source (4.20). Strictly speaking, when used with such external sources, all our previous integrals over \mathbf{p} should involve an ultraviolet cut-off of order gT .

To close this subsection, we return to the general case of an arbitrary non-abelian gauge field configuration (with $|A_\mu^a| \lesssim T$) and compute the total energy absorbed by the plasma during the life-time of the excited state:

$$\Delta\mathcal{W} \equiv \int_{-\infty}^{\infty} dt \frac{d\mathcal{W}}{dt} = \int_{-\infty}^{\infty} dt \int d^3x \mathbf{E}^a(t, \mathbf{x}) \cdot \mathbf{j}_a^{ind}(t, \mathbf{x}). \quad (4.21)$$

This quantity is well defined if we assume that the external perturbation is switched off adiabatically at $t \rightarrow \infty$. It coincides, of course, with the $t \rightarrow \infty$ limit of the polarization energy $\mathcal{W}(t)$ given by eq. (3.18). By using eq. (2.9) for \mathbf{j}_a^{ind} , we obtain, after some

elementary changes in the integration variables,

$$\Delta\mathcal{W} = \frac{3}{2}\omega_p^2 \int \frac{d\Omega}{4\pi} \int d^4x \mathbf{v} \cdot \mathbf{E}^a(x) \int_{-\infty}^{\infty} du U_{ab}(x, x-vu) \mathbf{v} \cdot \mathbf{E}^b(x-vu). \quad (4.22)$$

This quantity vanishes if

$$\int_{-\infty}^{\infty} du U_{ab}(x, x-vu) E_b^i(x-vu) = 0, \quad (4.23)$$

for arbitrary x , \mathbf{v} and i . For fixed \mathbf{v} , we may choose the axial gauge $v^\mu A_\mu^a = 0$. Then eq. (4.23) becomes

$$0 = \int_{-\infty}^{\infty} du E_a^i(x-vu) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \delta(p^0 - \mathbf{v} \cdot \mathbf{p}) E_a^i(p^0, \mathbf{p}). \quad (4.24)$$

As it should be valid for any x , eq. (4.24) implies $E_a^i(p^0 = \mathbf{v} \cdot \mathbf{p}, \mathbf{p}) = 0$, which is the condition for absence of Landau damping. More generally, eq. (4.23) characterizes the non-abelian gauge fields which propagate without dissipation. It is one of the conditions invoked in Ref.[1] in order to write a well-defined effective action for the soft gauge fields.

4.3 Plasma waves

We start by reconsidering the periodic field (4.14) and assume that $\mathbf{E}(\mathbf{p}) = 0$ if $|\mathbf{p}| > \omega_0$, so that there is no dissipation. The average, over the period T_0 , of the polarization energy $\mathcal{W}(t)$, eq. (3.17), is

$$\langle \mathcal{W} \rangle = \frac{3}{4} \pi \omega_p^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d\Omega}{4\pi} \frac{|\mathbf{v} \cdot \mathbf{E}(\mathbf{p})|^2}{(\omega_0 - \mathbf{v} \cdot \mathbf{p})^2}. \quad (4.25)$$

The angular integral can be easily evaluated (see eqs. (A.21) and (A.23)). After adding the Yang-Mills contribution,

$$\langle \mathcal{E}_{YM} \rangle = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left\{ |\mathbf{E}(\mathbf{p})|^2 + \frac{1}{\omega_0^2} |\mathbf{p} \times \mathbf{E}(\mathbf{p})|^2 \right\}, \quad (4.26)$$

we derive the following expression for the averaged value (over T_0) of the total field energy $\mathcal{E}(t) = \mathcal{E}_{YM}(t) + \mathcal{W}(t)$:

$$\langle \mathcal{E} \rangle = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left\{ \left[1 + 3 \frac{\omega_p^2}{p^2} a(\omega_0/p) \right] |\mathbf{E}_l(\mathbf{p})|^2 + \left[1 + \frac{p^2}{\omega_0^2} + 3 \frac{\omega_p^2}{p^2} b(\omega_0/p) \right] |\mathbf{E}_t(\mathbf{p})|^2 \right\}. \quad (4.27)$$

The coefficients $a(\omega_0/p)$ and $b(\omega_0/p)$ are given in eq. (A.23). They are both positive for $\omega_0/p > 1$. Furthermore, $\mathbf{E}(\mathbf{p}) \equiv \mathbf{E}_l(\mathbf{p}) + \mathbf{E}_t(\mathbf{p})$, with $\mathbf{p} \cdot \mathbf{E}_t(\mathbf{p}) = 0$.

The coefficients in eq. (4.27) can be related to the derivatives of the dielectric tensor (A.12) with respect to ω . Indeed, by using eqs. (A.14)–(A.15) for $\epsilon_{l,t}(\omega, p)$, it is straightforward to verify that eq. (4.27) coincides with

$$\langle \mathcal{E} \rangle = \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{d}{d\omega_0} [\omega_0 \epsilon_l(\omega_0, p)] |\mathbf{E}_l(\mathbf{p})|^2 + \frac{d}{d\omega_0} [\omega_0 (\epsilon_t(\omega_0, p) - p^2/\omega_0^2)] |\mathbf{E}_t(\mathbf{p})|^2 \right\}. \quad (4.28)$$

A similar expression holds for ordinary polarizable media[15].

The previous discussion can be easily generalized to study the *abelian plasma waves*, i.e., the solutions to eq. (4.1) without external sources. The decomposition of an arbitrary electric field in terms of normal modes is of the form

$$\mathbf{E}(t, \mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \sum_{s=l,t} \left\{ \mathbf{e}_s(\mathbf{p}) e^{-i\omega_s(p)t} + \mathbf{e}_s^*(-\mathbf{p}) e^{i\omega_s(p)t} \right\}, \quad (4.29)$$

where the frequencies $\omega_s(p)$ are solutions of the dispersion equations

$$\epsilon_l(\omega_l(p), p) = 0, \quad \text{and} \quad \epsilon_t(\omega_t(p), p) = p^2/\omega^2, \quad (4.30)$$

for longitudinal and transverse modes, respectively[9, 10, 20]. Note that $\omega_l(0) = \omega_t(0) = \omega_p$ and $\omega_{l,t}(p) > p$, for any p , and hence the normal modes propagate without damping.

By using the dispersion relations (4.30), and eqs. (A.17)–(A.18), we easily obtain the energy of the field (4.29) as

$$\begin{aligned} \mathcal{E} = \int \frac{d^3 p}{(2\pi)^3} & \left\{ \left(\frac{3\omega_p^2}{\omega_l^2(p) - p^2} - 1 \right) \mathbf{e}_l(\mathbf{p}) \cdot \mathbf{e}_l^*(\mathbf{p}) \right. \\ & \left. + \left(\frac{3\omega_p^2}{\omega_t^2(p) - p^2} + \frac{p^2}{\omega_t^2(p)} - 1 \right) \mathbf{e}_t(\mathbf{p}) \cdot \mathbf{e}_t^*(\mathbf{p}) \right\}. \end{aligned} \quad (4.31)$$

As expected, this is time-independent. The coefficients in eq. (4.31) are inversely proportional to the residues $z_{l,t}(p)$ of the effective propagators (A.4) at the appropriate poles (see eqs. (A.6) and (A.20)). Since the energy (4.31) is positive, the same holds for the residues $z_{l,t}(p)$; this can be also verified directly by using the formulae given in Appendix.

It is quite straightforward to verify that the second expression for the energy density, i.e. eq. (3.31), leads to the same excitation energy. More interestingly, eq. (3.30) allows us to compute easily the total 3-momentum of the field configuration (4.29). We obtain:

$$\begin{aligned} \mathbf{P} = \int \frac{d^3 p}{(2\pi)^3} & \left\{ \frac{\mathbf{p}}{\omega_l(p)} \left(\frac{3\omega_p^2}{\omega_l^2(p) - p^2} - 1 \right) \mathbf{e}_l(\mathbf{p}) \cdot \mathbf{e}_l^*(\mathbf{p}) \right. \\ & \left. + \frac{\mathbf{p}}{\omega_t(p)} \left(\frac{3\omega_p^2}{\omega_t^2(p) - p^2} + \frac{p^2}{\omega_t^2(p)} - 1 \right) \mathbf{e}_t(\mathbf{p}) \cdot \mathbf{e}_t^*(\mathbf{p}) \right\}. \end{aligned} \quad (4.32)$$

We have used here eq. (A.22) in Appendix, as well as the dispersion relations for the normal modes, eqs. (4.30).

Physically more transparent expressions for the energy and the momentum of the plasma waves are obtained by expressing them in terms of the gauge potentials which correspond to the field (4.29). We use the normal modes decomposition

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda=0}^2 \frac{z_\lambda^{1/2}(p)}{\sqrt{2\omega_\lambda(p)}} \left\{ \epsilon_\mu(\mathbf{p}; \lambda) a_\lambda(\mathbf{p}) e^{-ip \cdot x} + \epsilon_\mu^*(\mathbf{p}; \lambda) a_\lambda^*(\mathbf{p}) e^{ip \cdot x} \right\}, \quad (4.33)$$

where $\lambda = 0$ corresponds to the longitudinal mode, $\lambda = 1, 2$ to the transverse ones, $p^0 = \omega_\lambda(p)$, and the polarization vectors $\epsilon^\mu(\mathbf{p}; \lambda)$ are defined in Appendix. In terms of $a_\lambda(\mathbf{p})$ and $a_\lambda^*(\mathbf{p})$, the plasma wave energy and momentum become simply

$$\mathcal{E} = \int \frac{d^3p}{(2\pi)^3} \left\{ \omega_l(p) a_l(\mathbf{p}) a_l^*(\mathbf{p}) + \omega_t(p) \sum_{\lambda=1,2} a_\lambda(\mathbf{p}) a_\lambda^*(\mathbf{p}) \right\}, \quad (4.34)$$

and, respectively,

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \left\{ \mathbf{p} a_l(\mathbf{p}) a_l^*(\mathbf{p}) + \mathbf{p} \sum_{\lambda=1,2} a_\lambda(\mathbf{p}) a_\lambda^*(\mathbf{p}) \right\}. \quad (4.35)$$

The interpretation of the two equations above in terms of the color elementary oscillations is obvious.

5 Global color oscillations

In this section we study particular non-abelian solutions to the field equations which are uniform in space, i.e., which describe global color oscillations of the plasma. For convenience, we choose the temporal gauge, $A_a^0(x) = 0$. Therefore, $A_a^\mu(x) \equiv (0, \mathbf{A}^a(t))$. For such fields, it is useful to work with the following representation for $W_a^0(x; v)$:

$$W_a^0(x; v) = -\mathbf{v} \cdot \mathbf{A}^a(x) - \int_0^\infty du U_{ab}(x, x - vu) (\mathbf{v} \cdot \nabla_x) (v \cdot A^b(x - vu)). \quad (5.1)$$

This can be derived from eq. (4.3), by noting that $U(x, x - vu)$ satisfies the equation

$$\frac{\partial}{\partial u} U(x, x - vu) = -i g U(x, x - vu) v \cdot A(x - vu), \quad (5.2)$$

where $A^\mu \equiv A_a^\mu T^a$. For fields which do not depend on spatial coordinates, we have simply

$$W_a^0(t; v) = -\mathbf{v} \cdot \mathbf{A}^a(t). \quad (5.3)$$

From eq. (2.15), one obtains then $\rho_a^{ind}(t) = 0$ and $\mathbf{j}_a^{ind}(t) = -\omega_p^2 \mathbf{A}^a(t)$, and, using the definition (3.12),

$$\Theta^{00}(t) = \frac{1}{2} \omega_p^2 \mathbf{A}^a(t) \cdot \mathbf{A}^a(t). \quad (5.4)$$

This expression of Θ^{00} is very similar to that obtained in the static case, eq. (4.7); in particular, both eqs. (4.7) and (5.4) are local functionals of the fields. The color electric and magnetic fields are

$$\mathbf{E}^a(t) = -\frac{d\mathbf{A}^a}{dt} \quad \mathbf{B}^a(t) = \frac{g}{2} f^{abc} \mathbf{A}^b(t) \times \mathbf{A}^c(t). \quad (5.5)$$

Using these expressions, we obtain a simple expression for the total energy density in terms of the vector potentials

$$T^{00}(t) = \frac{1}{2} \left(\frac{d\mathbf{A}^a}{dt} \cdot \frac{d\mathbf{A}^a}{dt} + \omega_p^2 \mathbf{A}^a \cdot \mathbf{A}^a \right) + \frac{g^2}{4} f^{abc} f^{ade} (\mathbf{A}^b \cdot \mathbf{A}^d) (\mathbf{A}^c \cdot \mathbf{A}^e). \quad (5.6)$$

Eq. (5.4), together with the other components of the energy-momentum tensor, can also be obtained from $\Theta_{\text{sym}}^{\mu\nu}$ given in Sec.3.3. Indeed, for $A_a^\mu(x) = (0, \mathbf{A}^a(t))$, eqs. (3.39) and (3.42) imply

$$\Theta_{\text{sym}}^{\mu\nu} = 3\omega_p^2 \int \frac{d\Omega}{4\pi} v^\mu v^\nu \left\{ 2(\mathbf{v} \cdot \mathbf{A}^a)(\mathbf{v} \cdot \mathbf{A}^a) - \frac{1}{2} \mathbf{A}^a \cdot \mathbf{A}^a \right\}, \quad (5.7)$$

and it is easily verified that $\Theta_{\text{sym}}^{00} = \Theta^{00}$. Furthermore, $\Theta_{\text{sym}}^{0i} = 0$ and

$$\Theta_{\text{sym}}^{ij} = \omega_p^2 \left\{ \frac{4}{5} A_a^i A_a^j - \frac{\delta^{ij}}{10} \mathbf{A}^a \cdot \mathbf{A}^a \right\}. \quad (5.8)$$

The external color sources which are needed to generate the field configuration can be determined from the equations (2.1) which take here the form

$$\rho(t) = i g \left[A^i, \frac{dA^i}{dt} \right], \quad (5.9)$$

and

$$j^i(t) = \frac{d^2 A^i}{dt^2} + \omega_p^2 A^i + g^2 \left[[A^i, A^j], A^j \right]. \quad (5.10)$$

Note that these equations differ from the corresponding classical Yang-Mills equations in the vacuum only by the presence of a mass term $\sim \omega_p^2 A_a^i$ for the vector potential.

In what follows, we look for self-sustained global color oscillations, i.e., solutions to the equations (5.9)–(5.10) with vanishing external sources ($\rho(t) = j^i(t) = 0$). The

corresponding equations in the vacuum have been extensively investigated, mostly for the $SU(2)$ color group[21]. However, as we shall see, the presence of the mass term $\omega_p \sim gT$ dramatically affects the dynamical content of these equations. Besides, the respective solutions for the high temperature quark-gluon plasma have direct physical relevance, as they represent possible collective excitations of the system. The physical interpretation of the corresponding excitations in the confining regime is more obscure.

The condition $\rho(t) = 0$ implies, according to eq. (5.9), the commutativity of the color matrices $A^i(t) \equiv A_a^i(t)T^a$ and $\dot{A}^i(t) \equiv (dA_a^i/dt)T^a$. We shall consider here two non-trivial possibilities:

- (i) The vectors $\{A_a^i(t)\}$ and $\{\dot{A}_a^i(t)\}$ are parallel in color space for any i , which implies $A_a^i(t) = \mathcal{A}_a^i h^i(t)$ (no summation over i), with constant \mathcal{A}_a^i and arbitrary functions $h^i(t)$.
- (ii) The color matrices $A^i(t)$ belong to the Cartan subalgebra of $SU(N)$, $A^i(t) \equiv \sum_s A_s^i(t)T^s$. (Recall the discussion at the beginning of Sec.4.1.) If $N = 2$, (ii) is just a particular case of (i).

We consider now the consequences of the second set of equations, that is, $j^i(t) = 0$ for all i , on the two possibilities mentioned above.

- (i) For gauge potentials of the form $A^i(t) = \mathcal{A}^i h^i(t)$, ($\mathcal{A}^i \equiv \mathcal{A}_a^i T^a$), the normal modes equations $j^i = 0$ give generally non-linear and coupled second-order differential equations. As an example, consider $SU(2)$, where $f^{abc} = \epsilon^{abc}$, ($a = 1, 2, 3$), and the particular field configuration in which $\mathcal{A}_a^i = \delta_a^i$. Then, the functions $h_i(t)$ satisfy

$$\frac{d^2 h_1}{dt^2} + \omega_p^2 h_1 + g^2 h_1 (h_2^2 + h_3^2) = 0, \quad (5.11)$$

plus two similar equations for $h_2(t)$ and $h_3(t)$. The associated energy density,

$$T^{00} = \frac{1}{2} \sum_i \left(\left(\frac{dh_i}{dt} \right)^2 + \omega_p^2 h_i^2 \right) + \frac{g^2}{2} (h_1^2 h_2^2 + h_1^2 h_3^2 + h_2^2 h_3^2), \quad (5.12)$$

is an integral of the motion and acts as an effective Hamiltonian for the functions $h_i(t)$. Contrary to what happens at zero temperature, where $\omega_p = 0$, here the energy conservation prevents any trajectory $\{h_i(t)\}$ from getting too far away from the origin in the 3-dimensional space with axis h_i .

Periodic solutions to eqs. (5.11) can be easily written down in the particular case where all the three colors oscillate in phase: $h_1 = h_2 = h_3 \equiv h$ (“white oscillations”). The function $h(t)$ satisfies then the non-linear equation

$$\ddot{h} + \omega_p^2 h + 2g^2 h^3 = 0, \quad (5.13)$$

which has the following solution

$$h(t) = h_\theta \operatorname{cn}(\Omega_\theta(t - t_0); k), \quad (5.14)$$

for the initial conditions $h(t_0) = h_\theta$ and $\dot{h}(t_0) = 0$ (the overdots indicate time derivatives). Here, $\operatorname{cn}(x; k)$ is the Jacobi elliptic cosine of argument x and modulus k , and t_0 is the arbitrary origin of the time. The quantities k , h_θ and Ω_θ are related to the dimensionless parameter $\theta^2 \equiv (g^2/\omega_p^4)T^{00}$ by

$$k = \frac{1}{\sqrt{2}} \left[1 - \left(1 + \frac{8}{3} \theta^2 \right)^{-1/2} \right]^{1/2}, \quad (5.15)$$

$$h_\theta = \frac{\omega_p}{\sqrt{2}g} \left[\left(1 + \frac{8}{3} \theta^2 \right)^{1/2} - 1 \right]^{1/2}, \quad (5.16)$$

and

$$\Omega_\theta = \omega_p \left(1 + \frac{8}{3} \theta^2 \right)^{1/4}. \quad (5.17)$$

The solution (5.14) is periodic, with period

$$\mathcal{T}_\theta = \frac{4}{\Omega_\theta} K(k), \quad (5.18)$$

where $K(k)$ is the complete elliptic integral of modulus k . Since $|h(t)| \lesssim T$, then $\theta \lesssim 1$ and \mathcal{T}_θ is of order of $\mathcal{T}_0 \equiv 2\pi/\omega_p$. The associated field strenghts are $E_a^i = -\delta_a^i \dot{h}$, and $B_a^i = g h^2 \delta_a^i$. Therefore, the vectors \mathbf{E}_a and \mathbf{B}_a are parallel for any a .

Other non-abelian solutions to eqs. (5.11) are discussed in Ref. [22].

(ii) For gauge fields in the Cartan algebra, $A^i(t) \equiv \sum_s A_s^i(t) T^s$, the commutator entering eq. (5.10) is trivially zero, so that the normal mode equation reduces to

$$\frac{d^2 A_s^i}{dt^2} + \omega_p^2 A_s^i = 0. \quad (5.19)$$

Its general solution has the form

$$A_s^i(t) = \mathcal{A}_s^i \cos \omega_p t + \mathcal{B}_s^i \sin \omega_p t, \quad (5.20)$$

with arbitrary constants \mathcal{A}_s^i and \mathcal{B}_s^i . A particular solution of this type for $SU(3)$ is $\mathbf{A}(t) = T^3 \mathbf{a} \cos \omega_p t + T^8 \mathbf{b} \sin \omega_p t$, where \mathbf{a} and \mathbf{b} are fixed vectors in coordinate space. This describes coupled oscillations in both coordinate and color spaces, with an energy density

$$T^{00} = \frac{1}{2} \omega_p^2 (\mathbf{a}^2 + \mathbf{b}^2). \quad (5.21)$$

In particular, $a^i = b^i$ corresponds to global color rotations in the plane $T^3 - T^8$.

6 Conclusions

In this work we have studied the energy and the momentum of a general soft gauge field configuration describing collective excitations of the quark-gluon plasma. Our derivation relies on the explicit formulae that we have obtained previously for the induced current expressing the plasma response to the soft gauge fields. The simplicity of these expressions allowed us to construct three different, but physically equivalent, versions for the tensor $T^{\mu\nu}$, which are gauge-invariant and rather simple. One of these versions is symmetric (and traceless), which allowed us to simply obtain the gauge field angular momentum. We have shown that the total excitation energy is positive, for arbitrary gauge fields. We have also verified that, for particular abelian field configurations, our general expressions reduce to well-known formulae from classical plasma physics. Finally, we have analyzed the global color dynamics of the plasma and given a particular non-linear solution which describes a truly non-abelian excitation (the plasma white oscillations). The specific examples that we have considered in the last two sections illustrate, in particular, the utility of our general expressions for $T^{\mu\nu}$ for various applications.

Appendix

In this appendix we discuss the leading order effective propagator for soft gluons, together with some related topics, like the dielectric permittivity tensor for gauge plasmas and the polarization vectors for the gluonic normal modes.

As already discussed in Sec. 2, the dominant contribution to the polarization tensor for soft gauge fields is given by $\Pi_{\mu\nu}(P)$, eq. (2.12), and satisfies $P^\mu \Pi_{\mu\nu}(P) = 0$, consequence of the conservation law (2.10). The soft gluon propagator is therefore $*D_{\mu\nu}^{-1}(P) \equiv D_{0\mu\nu}^{-1}(P) + \Pi_{\mu\nu}(P)$ [11, 12]. In order to invert this expression, it is useful to decompose $\Pi_{\mu\nu}(P)$ into longitudinal and transverse components (relative to \mathbf{p}) [10]:

$$\Pi^{\mu\nu}(P) \equiv \Pi_l(p^0, p) \mathcal{P}_l^{\mu\nu} + \Pi_t(p^0, p) \mathcal{P}_t^{\mu\nu}, \quad (\text{A.1})$$

where the projection operators $\mathcal{P}_{l,t}^{\mu\nu}$ satisfy

$$\begin{aligned} P_\mu \mathcal{P}_l^{\mu\nu} &= P_\mu \mathcal{P}_t^{\mu\nu} = 0, & \mathcal{P}_l^{\mu\nu} + \mathcal{P}_t^{\mu\nu} &= (P^\mu P^\nu / P^2) - g^{\mu\nu}, \\ \mathcal{P}_l^{\mu\rho} \mathcal{P}_{l\rho\nu} &= -\mathcal{P}_{l\nu}^\mu, & \mathcal{P}_t^{\mu\rho} \mathcal{P}_{t\rho\nu} &= -\mathcal{P}_{t\nu}^\mu, & \mathcal{P}_l^{\mu\rho} \mathcal{P}_{t\rho\nu} &= 0, \end{aligned} \quad (\text{A.2})$$

with $P^2 = p_0^2 - \mathbf{p}^2$ and $\hat{p}^i = p^i/p$. The explicit forms of the functions $\Pi_{l,t}(p^0, p)$ follow easily from eqs. (A.1)–(A.2) and (2.12); they can be inferred from eqs. (A.14)–(A.15)

below. Then, the gluon propagator is

$${}^*D^{\mu\nu}(P) = -\mathcal{P}_l^{\mu\nu} {}^*\Delta_l(P) - \mathcal{P}_t^{\mu\nu} {}^*\Delta_t(P) + \frac{1}{\lambda} \frac{P^\mu P^\nu}{P^4}, \quad (\text{A.3})$$

in a covariant gauge with gauge fixing parameter λ ($\lambda = 1$ in Feynman gauge). Here

$${}^*\Delta_l^{-1}(P) \equiv P^2 - \Pi_l(p^0, p), \quad {}^*\Delta_t^{-1}(P) \equiv P^2 - \Pi_t(p^0, p), \quad (\text{A.4})$$

are the effective inverse propagators for longitudinal and, respectively, transverse gluons. They vanish on the mass-shell for soft quasi-gluons, thus defining the corresponding dispersion relations, which we denote by $\pm\omega_s(p)$, with $s = l$ or t . The mass-shell residues are defined by the relation

$${}^*\Delta_s(\omega, p) \equiv \frac{1}{\omega^2 - p^2 - \Pi_s(\omega, p)} \approx \frac{z_s(p)}{\omega^2 - \omega_s^2(p)}, \quad (\text{A.5})$$

where the approximate equality holds in the vicinity of the pole. It follows that

$$z_s^{-1}(p) = \frac{1}{2\omega_s(p)} \frac{d}{d\omega} {}^*\Delta_s^{-1}(\omega, p) \Big|_{\omega_s(p)}. \quad (\text{A.6})$$

As discussed in Sec.4.3, after eq. (4.31), the residues $z_s(p)$ are positive functions.

Let us construct now covariant gauge polarization vectors appropriate for the longitudinal ($\epsilon^\mu(\mathbf{p}; 0)$), and, respectively, for the transverse ($\epsilon^\mu(\mathbf{p}; \lambda)$, $\lambda = 1, 2$) soft normal modes. They satisfy

$${}^*D_{\mu\nu}^{-1}(\omega_\lambda(p), \mathbf{p}) \epsilon^\nu(\mathbf{p}; \lambda) = 0, \quad (\text{A.7})$$

where $\omega_\lambda(p)$ is equal to $\omega_l(p)$ for $\lambda = 0$ and, respectively, to $\omega_t(p)$ for $\lambda = 1, 2$. By writting $\epsilon^\mu(\mathbf{p}; \lambda) = (0, \boldsymbol{\epsilon}(\mathbf{p}; \lambda))$, it follows that $P_\mu \epsilon^\mu(\mathbf{p}; \lambda) = 0$, for $\lambda = 0, 1, 2$ and $p^0 = \omega_\lambda(p)$; furthermore, $\mathbf{p} \cdot \boldsymbol{\epsilon}(\mathbf{p}; \lambda) = 0$, for $\lambda = 1, 2$, and $(\delta^{ij} - \hat{p}^i \hat{p}^j) \epsilon^j(\mathbf{p}; 0) = 0$. With the normalization $\epsilon(\mathbf{p}; \lambda) \cdot \epsilon^*(\mathbf{p}; \lambda') = -\delta_{\lambda\lambda'}$, we obtain

$$\epsilon^\mu(\mathbf{p}; 0) = \beta(p) (1, \hat{\mathbf{p}} \omega_l/p), \quad \beta(p) \beta^*(p) = p^2/(\omega_l^2 - p^2), \quad (\text{A.8})$$

and

$$\epsilon^\mu(\mathbf{p}; \lambda) = (0, \boldsymbol{\epsilon}(\mathbf{p}; \lambda)), \quad \mathbf{p} \cdot \boldsymbol{\epsilon}(\mathbf{p}; \lambda) = 0, \quad \boldsymbol{\epsilon}(\mathbf{p}; \lambda) \cdot \boldsymbol{\epsilon}^*(\mathbf{p}; \lambda') = \delta_{\lambda\lambda'}, \quad (\text{A.9})$$

for $\lambda = 1, 2$. Note also that

$$\sum_{\lambda=1,2} \epsilon^\mu(\mathbf{p}; \lambda) \epsilon^{\nu*}(\mathbf{p}; \lambda) = \mathcal{P}_t^{\mu\nu}, \quad \epsilon^\mu(\mathbf{p}; 0) \epsilon^{\nu*}(\mathbf{p}; 0) = \mathcal{P}_l^{\mu\nu}. \quad (\text{A.10})$$

The soft gluon effective propagator (A.3) is intimately related to the *dielectric permittivity tensor*, commonly used in classical plasma physics to describe the polarization properties of the plasma. In the weak field (or the abelian) limit, where the plasma responds linearly, (i.e., $j_\mu^{ind} = \Pi_{\mu\nu} A^\nu$), we can use the standard definition of ϵ^{ij} [16]. The *polarization vector* $\mathbf{P}(\omega, \mathbf{p})$ satisfies, by definition, $-i\omega \mathbf{P}(\omega, \mathbf{p}) \equiv \mathbf{j}^{ind}(\omega, \mathbf{p})$, and also $i\mathbf{p} \cdot \mathbf{P}(\omega, \mathbf{p}) = -\rho^{ind}(\omega, \mathbf{p})$, because of the continuity equation for j_μ^{ind} , eq. (2.10). The linear relation between the *displacement vector* $\mathbf{D}(\omega, \mathbf{p}) \equiv \mathbf{E}(\omega, \mathbf{p}) + \mathbf{P}(\omega, \mathbf{p})$ and the electric field defines the dielectric tensor:

$$D^i(\omega, \mathbf{p}) \equiv \epsilon^{ij}(\omega, \mathbf{p}) E^j(\omega, \mathbf{p}). \quad (\text{A.11})$$

It follows immediately that

$$\epsilon^{ij}(\omega, \mathbf{p}) = \delta^{ij} - \frac{1}{\omega^2} \Pi^{ij}(\omega, \mathbf{p}). \quad (\text{A.12})$$

For isotropic dielectric media, ϵ^{ij} has only two independent components, and it is useful to choose them as the longitudinal and the transverse dielectric functions, defined by

$$\epsilon^{ij}(\omega, \mathbf{p}) = \epsilon_l(\omega, p) \hat{p}^i \hat{p}^j + \epsilon_t(\omega, p) (\delta^{ij} - \hat{p}^i \hat{p}^j). \quad (\text{A.13})$$

From this relation, together with eqs. (A.12) and (2.12), it follows that

$$\epsilon_l(\omega, p) = 1 - \frac{1}{\omega^2 - p^2} \Pi_l(\omega, p) = 1 - 3 \frac{\omega_p^2}{p^2} [Q(\omega/p) - 1], \quad (\text{A.14})$$

$$\epsilon_t(\omega, p) = 1 - \frac{1}{\omega^2} \Pi_t(\omega, p) = 1 - \frac{3}{2} \frac{\omega_p^2}{p^2} \left[1 - \frac{\omega^2 - p^2}{\omega^2} Q(\omega/p) \right], \quad (\text{A.15})$$

where

$$Q(x) \equiv \frac{x}{2} \ln \frac{x+1}{x-1} = \frac{x}{2} \left(\ln \left| \frac{x+1}{x-1} \right| - i\pi\theta(1-|x|) \right), \quad (\text{A.16})$$

and the prescription $x \rightarrow x + i\eta$, $\eta \rightarrow 0_+$, has been understood in writing the imaginary part of $Q(x)$. Remark the imaginary parts in $\epsilon_{l,t}(\omega, p)$, which occur for $|\omega| < p$, as expected. From eqs. (A.14), (A.15) and (4.30), we derive

$$\frac{d}{d\omega} (\omega \epsilon_l(\omega, p)) \Big|_{\omega_l(p)} = \omega_l(p) \frac{d\epsilon_l}{d\omega} \Big|_{\omega_l(p)} = \frac{3\omega_p^2}{\omega_l^2(p) - p^2} - 1, \quad (\text{A.17})$$

and, similarly,

$$\begin{aligned} \frac{d}{d\omega} [\omega (\epsilon_t(\omega, p) - p^2/\omega^2)] \Big|_{\omega_t(p)} &= \omega_t(p) \frac{d}{d\omega} (\epsilon_t(\omega, p) - p^2/\omega^2) \Big|_{\omega_t(p)} \\ &= \frac{3\omega_p^2}{\omega_t^2(p) - p^2} + \frac{p^2}{\omega_t^2(p)} - 1. \end{aligned} \quad (\text{A.18})$$

The dielectric functions are related to the inverse effective propagators, as shown by eqs. (A.4) and (A.14)–(A.15),

$${}^*\Delta_l^{-1}(\omega, p) = (\omega^2 - p^2) \epsilon_l(\omega, p), \quad {}^*\Delta_t^{-1}(\omega, p) = \omega^2 \epsilon_t(\omega, p) - p^2. \quad (\text{A.19})$$

From the defining relation of the residues, eq. (A.6), it follows that

$$\begin{aligned} \omega_l(p) \frac{d\epsilon_l}{d\omega} \Big|_{\omega_l(p)} &= \frac{2\omega_l^2(p)}{\omega_l^2(p) - p^2} z_l^{-1}(p), \\ \omega_t(p) \frac{d}{d\omega} \left(\epsilon_t(\omega, p) - p^2/\omega^2 \right) \Big|_{\omega_t(p)} &= 2 z_t^{-1}(p). \end{aligned} \quad (\text{A.20})$$

Note finally the following angular integrals, which are needed in Sec.4.3 in order to compute the energy and the momentum of the plasma waves:

$$\int \frac{d\Omega}{4\pi} \frac{v^i v^j}{(\omega - \mathbf{v} \cdot \mathbf{p})^2} = \frac{a(\omega/p)}{p^2} \hat{p}^i \hat{p}^j + \frac{b(\omega/p)}{p^2} (\delta^{ij} - \hat{p}^i \hat{p}^j), \quad (\text{A.21})$$

and

$$\int \frac{d\Omega}{4\pi} \frac{v^i v^j}{(\omega - \mathbf{v} \cdot \mathbf{p})^3} = \frac{c(\omega/p)}{p^3} \hat{p}^i \hat{p}^j + \frac{d(\omega/p)}{p^3} (\delta^{ij} - \hat{p}^i \hat{p}^j). \quad (\text{A.22})$$

These results are valid for $\omega > p$. In these formulae

$$a(x) \equiv 1 + \frac{x^2}{x^2 - 1} - 2Q(x), \quad b(x) \equiv Q(x) - 1, \quad (\text{A.23})$$

and

$$c(x) \equiv \frac{1}{x} \left\{ Q(x) - \left(1 - \frac{1}{x^2 - 1} \right) \frac{x^2}{x^2 - 1} \right\}, \quad d(x) \equiv \frac{1}{2x} \left(\frac{x^2}{x^2 - 1} - Q(x) \right), \quad (\text{A.24})$$

with $Q(x)$ given by (A.16). For $x > 1$, the functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are all positive and vanish only in the limit $x \rightarrow \infty$.

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